# HEAT TRANSFER *in* COMPOSITE MATERIALS

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**Heat Transfer in Composite Materials**

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# Contents

# Preface

THE use of composite materials in industry began in the 1940s with GFRP (glass fiber reinforced plastics). Since then, the research and development of composite materials have gone through significant breakthroughs turning the subject of composite materials into a matured discipline in applied science today. The most compelling reason for industrial use of composite materials is to take advantage of their anisotropic nature which enables such material properties as strength, stiffness and fracture toughness to be tailored to specific directions.

Composites often fail when they are subject to severe environments such as high temperatures or cryogenic temperatures even though there is no external load applied to the composites. Typically, failure in composites is caused by thermal stresses which are generated at the interface between the reinforcing fibers and the surrounding matrix due to the mismatch of the thermal expansion coefficients of the two materials. The mismatch of thermal expansion coefficients is not the only cause for thermal stresses. Thermal stresses are also generated by the mismatch of the thermal conductivities and the elastic stiffness if the temperature distribution in the composite is non-uniform.

In order to understand how the thermal stress is generated and distributed in the composites, it is necessary to know the temperature distribution in the composite first. It is also important to know the effective thermal conductivity of the composite when it is viewed as an equivalent homogeneous medium that exhibits the same response as the composite. While the mechanical behavior of composite materials has been investigated for decades, research on the thermal properties of composites is somewhat less explored, which motivated the authors to write this book.

One of the authors (SN) has been working on developing analytical methods for the mechanical properties of composites using micromechanics approaches in which microstructures such as inclusions, defects and dislocations are taken into account that affect the properties of heterogeneous materials. The other author (AHS) has been working on a wide range of heat transfer research and developed semi-analytical methods to accurately predict the temperature distribution in anisotropic media for both steady-state and transient state. Therefore it was logical that both of us decided to write a book on analytical methods for heat transfer in composites where the availability of books on this topic was scarce in the composite research community. Steady-state heat transfer in composites composed of inclusions (fibers) and a matrix phase of both finite and infinite size as well as the thermal stress resulting from the temperature distribution were handled by SN and both steady-state and transient-state heat transfer for muli-layer composites was handled by AHS.

The purpose of this book is to introduce analytical methods that can be used to obtain temperature distributions in general heterogeneous materials. Although many heat transfer problems in composites are routinely solved by numerical methods such as the finite element method or the finite difference method, analytical solutions are always preferred to numerical solutions. The readers are expected to have basic mathematical background at the undergraduate level of vector calculus, linear algebra and differential equations. As many of the formulas are lengthy and tedious, it is helpful if the readers have access to computer algebra software such as Mathematica and Maple that automates symbolic derivations of many mathematical formulas. For steady-state heat conduction, the governing equation for heat conduction is similar to the equations for permeability, electrical conductivity, diffusivity, dielectric constant, and magnetic permeability. However, the transient behavior of heat conduction significantly differs from the properties above which is handled in Chapter 4.

The book consists of seven Chapters. Chapter 1 is a general introduction and review of the heat conduction equations. Exemplar problems are solved that represent the basics of analytical methods employed in the subsequent Chapters. In Chapter 2, steady-state heat conduction in composites reinforced by fibers/particulates is solved from the micromechanics viewpoint. The temperature distribution in a medium that contains a spheroidal inclusion is sought and it is expressed analytically for an unbounded medium and semi-analytically for a bounded medium. A spheroidal inclusion can cover a wide range of fiber shapes from a flat-flake to a sphere to a long fiber. Chapter 3 discusses the analytical solutions for steady-state heat conduction in laminated multilayer composites and in heterogeneous materials. In Chapter 4, different analytical solutions for transient heat conduction in multilayer and laminated composites are emphasized. Chapter 5 presents an overview of rapid energy transport in heterogeneous composites under a local thermal non-equilibrium condition. Chapter 6 discusses the effective thermal conductivity of unbounded composite materials by introducing critical theoretical models including the upper and lower bounds of Hashin and Shtrikman, the Maxwell-Garnett effective medium theory, the Mori-Tanaka model and the self-consistent approximation. Chapter 7 discusses thermal stresses caused by a mismatch of the thermal expansion coefficients at the interface of the matrix and fibers due to a non-uniform temperature distribution in the composites. Although thermal stress analysis is not a subject within heat transfer, it is an important topic as the composites often fail due to non-uniform temperature distributions. The thermal stress analysis is carried out based on the results from the preceding Chapters. References are given at the end of each Chapter.

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CHAPTER 1

# Basic Equations for Heat Transfer

## 1.1. FOURIER'S LAW

In this Chapter, the fundamental equations of heat transfer are derived and well-known exemplar problems are solved. As this book is not meant to be a general textbook on heat transfer, the readers are referred to one of many outstanding textbooks (e.g. [1]) and only a minimum amount of equations to be used in the subsequent Chapters are presented. However, the formulations are intended to be applicable to anisotropic and heterogeneous materials later that define the composite materials. Although a preferred way of describing the mechanical and physical properties of anisotropic materials is to use the tensor (index) notation, it is not employed in this book except for Chapter 7 to avoid unnecessary complexity. More discussion on heat conduction in composites using index notation is found in [2].

The heat conduction equation is derived from Fourier's law. Fourier's law is an empirical relationship between the heat flow and temperature gradient. Fourier's law states that the heat flows when there is a temperature difference between two points (Figure 1.1). The amount of the flow is proportional to the temperature difference and the direction of the flow is also related to the direction of the temperature difference, which can be expressed as

$$\mathbf{q} \propto \nabla T \tag{1.1}$$

where **q** is the heat flux (a vector) whose dimension is w/m$^2$, $T$ is the

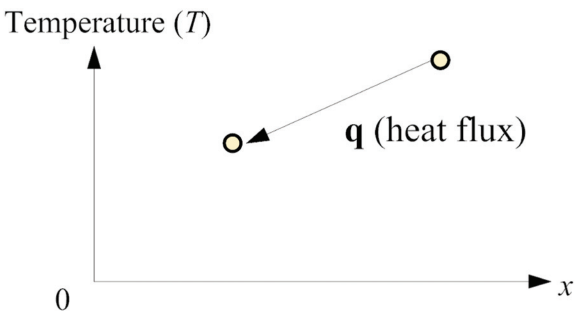


***FIGURE 1.1.*** *Fourier's law stating that heat flows from a point of higher temperature to a point of lower temperature.*

temperature (a scalar) and $\nabla T$ (a vector) is the gradient of $T$. Mathematically, Equation (1.1) can be expressed as

$$\mathbf{q} = -K\nabla T \tag{1.2}$$

where $K$ is a 3 × 3 matrix which is the proportionality factor between the heat flux, **q**, and the temperature gradient, $\nabla T$, and is called thermal conductivity with the dimension, w/(m·k). Equation (1.2) is explicitly expressed as

$$\begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = -\begin{pmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{pmatrix} \begin{pmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \\ \dfrac{\partial T}{\partial z} \end{pmatrix} \tag{1.3}$$

The minus sign in Equation (1.2) comes from the fact that heat flows from a point of higher temperature to a point of lower temperature. It should be noted that according to Onsager's principle [3], $K$ is symmetrical as

$$k_{xy} = k_{yx}, \quad k_{yz} = k_{zy}, \quad k_{zx} = k_{xz} \tag{1.4}$$

Hence, the number of independent components for $K$ is 6 in 3-D (4 in 2-D).

The general definition of composite materials is any medium which is heterogeneous. However, in engineering, composite materials are referred to those materials whose physical and mechanical properties are piecewise constant across different phases. If the material is isotropic, i.e., the properties are unchanged under rotations and reflections, the thermal conductivity, $K$, is represented by a single component, $k$, and Equation (1.3) is reduced to

$$\mathbf{q} = -k\nabla T \tag{1.5}$$

or

$$\begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = -k \begin{pmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \\ \dfrac{\partial T}{\partial z} \end{pmatrix} \tag{1.6}$$

When the material is orthotropic, i.e., the material properties are symmetrical with respect to the $x$-$y$, $y$-$z$ and $z$-x planes, the thermal conductivity, $K$, is expressed as a diagonal matrix as

$$K = \begin{pmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{pmatrix} \tag{1.7}$$

***TABLE 1.1. Thermal Conductivity for Common Materials [4].***

| Material | Thermal Conductivity (W/(m/K)) at 25°C |
|---|---|
| Air | 0.024 |
| Aluminum | 205 |
| Boron | 27 |
| Carbon | 1.7 |
| Cast iron | 58 |
| Concrete (lightweight) | 0.1–0.3 |
| Copper | 401 |
| Epoxy | 0.35 |
| Fiberglass | 0.04 |
| Glass | 1.05 |
| Polyester | 0.05 |
| Silicon carbide | 15.2 |
| Titanium | 22 |

In general, as the matrix, $K$, is symmetrical, it is always possible to rotate the coordinate system so that $K$ in Equation (1.3) is reduced to Equation (1.7). Table 1.1 lists thermal conductivities for typical materials taken from a freely available website, *engineeringtoolbox.com* [4].

## 1.2. EQUATION OF ENERGY

The heat conduction equation is derived based on the principle of energy balance. The first law of thermodynamics states that the rate change of energy in the body without motion is caused by the amount of heat entering the system and a heat source generated within the system, which can be expressed as (see Figure 1.2)

$$\int_{\Omega} \frac{\partial}{\partial t}(\rho E) dv = -\oint_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} ds + \int_{\Omega} g dv \tag{1.8}$$

where $\rho$ is the mass density, $E$ is the internal energy, $\mathbf{q}$ is the heat flux, $g$ is the internal heat source and $\mathbf{n}$ is the normal to the material boundary. The symbol $\Omega$ is the entire volume and $\partial\Omega$ is its boundary. The volume element is denoted as $dv$ and the surface element is denoted as $ds$.

Using the Gauss theorem[1], the first term in the right hand side of Equation (1.8) can be written from the boundary integral to the volume integral as

$$\oint_{\partial\Omega} -\mathbf{q} \cdot \mathbf{n}\, ds = -\int_{\Omega} \nabla \cdot \mathbf{q}\, dv \tag{1.9}$$

Therefore, Equation (1.8) becomes

$$\int_{\Omega} \frac{\partial}{\partial t}(\rho E) dv = -\int_{\Omega} \nabla \cdot \mathbf{q}\, dv + \int_{\Omega} g\, dv \tag{1.10}$$

or

$$\frac{\partial}{\partial t}(\rho E) = -\nabla \cdot \mathbf{q} + g \tag{1.11}$$

[1]The Gauss theorem is stated as

$$\oint_{\partial\Omega} \mathbf{n} \cdot \mathbf{v} ds = \int_{\Omega} \nabla \cdot \mathbf{v} dv$$

where the integral on the left hand side is an integral over the boundary of the body and the integral in the right is over the body. The quantity, $\mathbf{n}$, is the normal to the boundary. It is the 2- and 3-D versions of the *fundamental theorem of calculus* that states that integration and differentiation are reciprocal each other.

which is called the *equation of energy*. If the body is not in motion, it follows that $E = C_p T$ where $C_p$ is the specific heat. Using Fourier's law of Equation (1.2), Equation (1.11) can be written as

$$\rho C_p \frac{\partial T}{\partial t} = \nabla \cdot (K \nabla T) + g \tag{1.12}$$

For steady-state heat conduction, Equation (1.12) is reduced to

$$\nabla \cdot (K \nabla T) + g = 0 \tag{1.13}$$

If the medium is homogeneous and isotropic, the steady-state heat conduction equation is further reduced to the Poisson equation expressed as

$$k \Delta T + g = 0 \tag{1.14}$$

where $\Delta$ is the Laplace operator defined as

$$\Delta T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \tag{1.15}$$

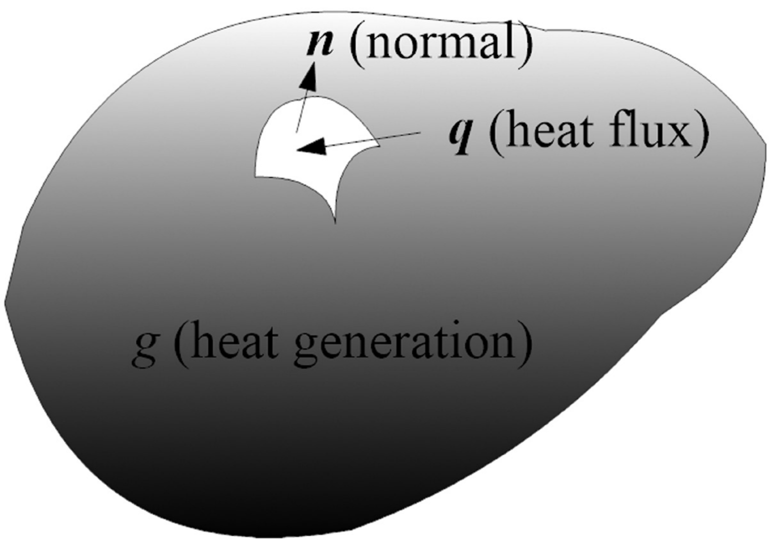


***FIGURE 1.2.*** *Heat flux,* **q**, *entering the body and heat generation, g, inside the body.*

## 1.3. EXAMPLES OF TEMPERATURE DISTRIBUTION IN HOMOGENEOUS MATERIALS

In this section, three typical problems of heat conduction in homogeneous materials are solved analytically. Although these examples are not directly related to composite materials, the employed analytical method is to be used for composite materials after appropriate modifications as shown in the subsequent Chapters. These examples are a few of the rare cases where the solutions for the temperature distribution can be expressed analytically in series form. It is noted that most of the heat conduction problems of importance in composites have no analytical solutions available.

### 1.3.1. Steady-State Temperature Distribution in a Square with a Heat Source

In this example, a 2-D square plate is considered in which a heat source, $g$, exists inside with the homogeneous boundary condition as shown in Figure 1.3. In steady-state, the heat conduction equation with a homogeneous boundary condition becomes a Poisson type differential equation as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + c = 0 \quad \text{in} \quad D \tag{1.16}$$

$$T = 0 \quad \text{on} \quad \partial D \tag{1.17}$$

where $T$ is the temperature, $\Omega$ is the domain, $\partial\Omega$ is its boundary and

$$c = \frac{g}{k} \tag{1.18}$$

is a function of $x$ and $y$.

As is shown in the subsequent Chapters, one of the powerful analytical methods that can be used for a variety of boundary value problems is the Sturm-Liouville approach [5]. In the Sturm-Liouville system corresponding to Equation (1.16), the eigenfunctions, $e_{mn}$, and the eigenvalues, $\lambda_{mn}$, are defined as the solution to the following differential equation

$$\frac{\partial^2 e_{mn}}{\partial x^2} + \frac{\partial^2 e_{mn}}{\partial y^2} + \lambda_{mn} e_{mn} = 0 \tag{1.19}$$

along with the homogeneous boundary condition:

$$e_{mn} = 0 \quad \text{on} \quad \partial\Omega \tag{1.20}$$

Equation (1.19) can be solved as

$$e_{mn} = 2 \sin m\pi x \sin n\pi y \tag{1.21}$$

and

$$\lambda_{mn} = \pi^2 (m^2 + n^2) \tag{1.22}$$

The function, $e_{mn}$, is called the eigenfunction and $\lambda_{mn}$ is called the eigenvalue for Equation (1.19). Note that $e_{mn}$ are orthogonal each other as

$$\int\int_{\Omega} e_{mn} e_{m'n'}\, dxdy = \delta_{mm'}\, \delta_{mn'} \tag{1.23}$$

where $\delta_{mn}$ is the Kronecker delta that returns 1 when $m = n$ and 0 otherwise and the integral range is over the entire square. A set of eigenfunctions forms the bases for a function that belong to the same linear

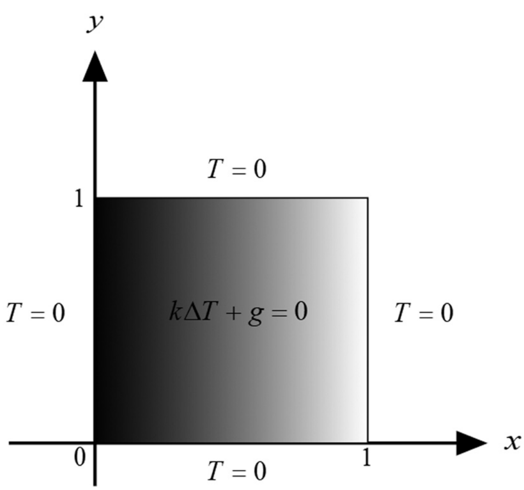


*FIGURE 1.3. A square plate having a uniform heat source with the homogeneous boundary condition.*

space (Hilbert space) as the eigenfunctions. Therefore, the temperature, $T(x, y)$, can be expressed as

$$T(x,y) = \sum_{m=1,n=1}^{\infty} T_{mn} e_{mn}(x,y) \tag{1.24}$$

where $T_{mn}$ is an unknown coefficient yet to be determined. The function, $c$, in Equation (1.16) can be also expanded by the eigenfunction, $e_{mn}$, as

$$c(x,y) = \sum_{m=1,n=1}^{\infty} c_{mn} e_{mn}(x,y) \tag{1.25}$$

where the expansion coefficient, $c_{mn}$, can be expressed as

$$c_{mn} = \int\int_{\Omega} c(x,y) e_{mn}(x,y) dxdy \tag{1.26}$$

Substitution of Equations. (1.24) and (1.25) into Equation (1.16) yields

$$\sum_{m=1,n=1}^{\infty} T_{mn} \pi^2 (m^2 + n^2) e_{mn} = \sum_{m=1,n=1}^{\infty} c_{mn} e_{mn} \tag{1.27}$$

from which $T_m$ can be obtained as

$$T_{mn} = \frac{c_{mn}}{\pi^2 (m^2 + n^2)} \tag{1.28}$$

Therefore, the temperature is expressed as

$$T(x,y) = \sum_{m=1,n=1}^{\infty} \frac{2c_{mn}}{\pi^2 (m^2 + n^2)} \sin m\pi x \sin n\pi y \tag{1.29}$$

For example, if the heat source is a unity,

$$c = 1$$

the expansion coefficient for $c$ can be computed as

$$c_{mn} = \frac{2((-1)^m - 1)((-1)^n - 1)}{\pi^2 mn} \tag{1.30}$$

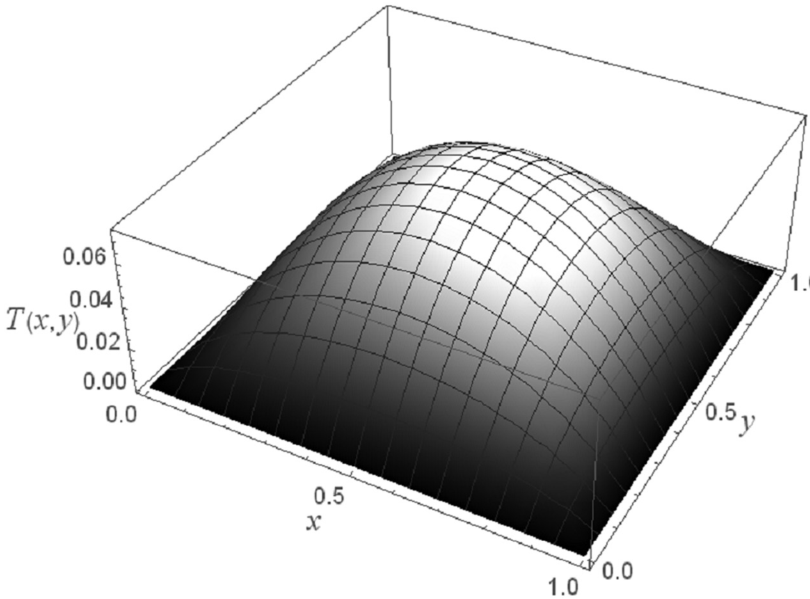


***FIGURE 1.4.*** *Temperature profile, T(x, y), over a square plate. Heat source, g = 1 with the homogeneous boundary condition.*

Therefore, the temperature is expressed as

$$T(x,y) = \sum_{m=1,n=1}^{\infty} \frac{4((-1)^m - 1)((-1)^n - 1)}{\pi^4 mn(m^2 + n^2)} \sin m\pi x \sin n\pi y \quad (1.31)$$

Figure 1.4 shows the profile of $T(x, y)$. The concept of expanding the temperature with a linear combination of eigenfunctions will be used in the subsequent Chapters.

### 1.3.2. Steady-State Temperature Distribution in a Square Plate with Non-homogeneous Boundary Condition

The example in Section 1.3.1 is for a square plate having a heat source with the homogeneous boundary condition. In this example, a similar problem with a non-homogeneous boundary condition having no heat source is solved with the boundary condition shown in Figure 1.5. The boundary condition is the first kind (the Dirichlet type) and

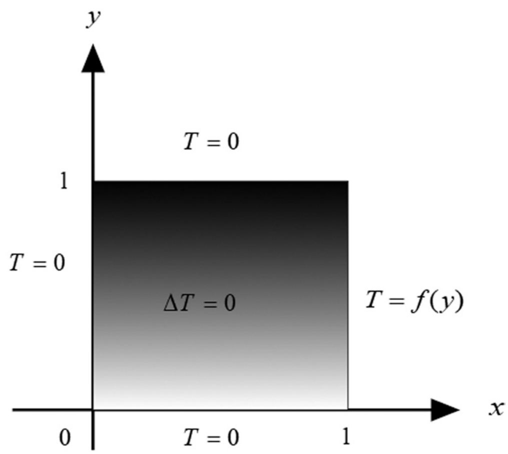


***FIGURE 1.5.*** *Steady-state temperature in a square without heat source but with a non-homogeneous boundary condition.*

the temperature is prescribed along the four sides. The differential equation with the prescribed boundary conditions is expressed as

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \ \text{ in } \ D \tag{1.32}$$

$$T(0,y)=0 \quad (0<y<1)$$
$$T(1,y)=f(y) \quad (0<y<1)$$
$$T(x,0)=T(x,1)=0 \quad (0<x<1)$$

The boundary condition is homogeneous in the $y$ direction but not in the $x$ direction. Therefore, the Sturm-Liouville system is applied in the $y$ direction only and Equation (1.32) can be rewritten as

$$L_y T=\frac{\partial^2 T}{\partial x^2} \tag{1.33}$$

where

$$L_y \equiv -\frac{\partial^2}{\partial y^2} \tag{1.34}$$

with the boundary conditions that $T = 0$ along $y = 0{,}1$ and $T = f(y)$ along $x = 1$. Because of the homogeneous boundary condition in the

$y$ direction, the variable, $y$, is chosen as the primary variable and $x$ as the secondary variable. Thus, the solution to Equation (1.33) is sought in a format of

$$T(x,y) = \sum_{n=1}^{\infty} T_n e_n(y) \tag{1.35}$$

where

$$e_n(y) = \sqrt{2}\sin n\pi y \tag{1.36}$$

$$\lambda_n = n^2\pi^2 \tag{1.37}$$

The function, $e_n(y)$, and $\lambda_n$, are the eigenfunctions and the eigenvalue for $L_y e_n = \lambda_n e_n$. The expansion coefficient, $T_n$, is independent of $y$ but dependent on $x$. Equation (1.35) is substituted into Equation (1.33) to yield

$$L_y\left(\sum_{n=1}^{\infty} T_n e_n(y)\right) = \frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} T_n e_n(y)\right)$$

$$\sum_{n=1}^{\infty} T_n \lambda_n e_n(y) = \sum_{n=1}^{\infty}\left(\frac{\partial^2 T_n}{\partial x^2}\right) e_n(y) \tag{1.38}$$

or

$$\frac{\partial^2 T_n}{\partial x^2} - \lambda_n T_n = 0 \tag{1.39}$$

which is solved as

$$T_n = A_n e^{\sqrt{\lambda_n}x} + B_n e^{-\sqrt{\lambda_n}x} \tag{1.40}$$

Thus, $T(x, y)$ can be expressed as

$$T(x,y) = \sum_{n=1}^{\infty} (A_n e^{\sqrt{\lambda_n}x} + B_n e^{-\sqrt{\lambda_n}x}) e_n(y) \tag{1.41}$$

where $A_n$ and $B_n$ are integral constants yet to be determined. The boundary condition at $x = 0$ can be used to yield

$$0 = \sum_{n=1}^{\infty} (A_n + B_n) e_n(y) \tag{1.42}$$

implying that $B_n = -A_n$, which reduces Equation (1.41) to

$$T(x, y) = \sum_{n=1}^{\infty} A_n (e^{\sqrt{\lambda_n} x} - e^{-\sqrt{\lambda_n} x}) e_n(y) \tag{1.43}$$

At $x = 1$,

$$f(y) = \sum_{n=1}^{\infty} A_n (e^{\sqrt{\lambda_n}} - e^{-\sqrt{\lambda_n}}) e_n(y) \tag{1.44}$$

which implies that

$$A_n = \frac{(f, e_n)}{e^{\sqrt{\lambda_n}} - e^{-\sqrt{\lambda_n}}} \tag{1.45}$$

Finally, the temperature, $T(x, y)$, is expressed as

$$T(x, y) = \sum_{n=1}^{\infty} \frac{(f, e_n)}{e^{\sqrt{\lambda_n}} - e^{-\sqrt{\lambda_n}}} (e^{\sqrt{\lambda_n} x} - e^{-\sqrt{\lambda_n} x}) e_n(y) \tag{1.46}$$

Figure 1.6 is an example of the profile of $T(x, y)$ when $f(y) = 1$.

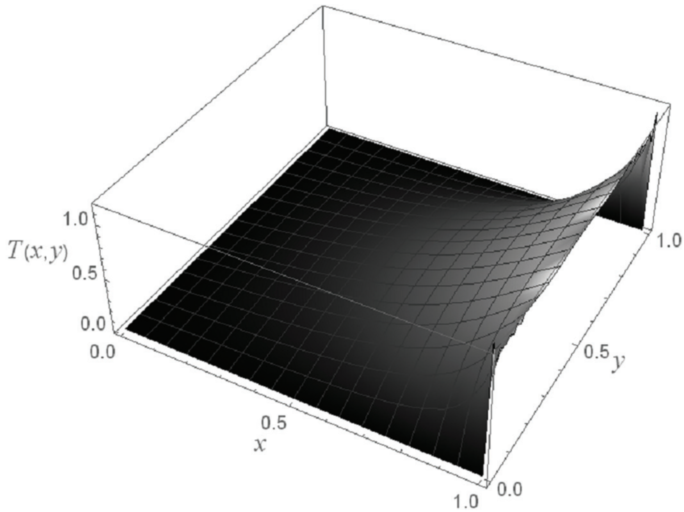


***FIGURE 1.6.*** *Temperature distribution in a square without heat source but with a non-homogeneous boundary condition.*

### 1.3.3. 1-D Transient Heat Conduction

In this example, a classical 1-D transient heat conduction equation is solved. The non-dimensionalized heat conduction equation is expressed as

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{\partial T(x,t)}{\partial t} \tag{1.47}$$

or

$$LT(x,t) = -\frac{\partial T(x,t)}{\partial t} \tag{1.48}$$

where $L$ is defined as

$$L \equiv -\frac{\partial^2}{\partial x^2} \tag{1.49}$$

The initial and boundary conditions are

- Initial condition: $T(x, 0) = f(\mathrm{x}) \quad (0 < x < 1)$
- Boundary condition: $T(0, t) = T(1, t) = 0 \quad (t \geq 0)$

For the differential operator, $L$, its accompanying eigenfunctions and eigenvalues defined as

$$\begin{gathered} Le_n = \lambda_n e_n \\ e_n(0) = e_n(1) = 0 \end{gathered} \tag{1.50}$$

are expressed as

$$e_n = \sqrt{2}\sin\sqrt{\lambda_n}\,x \tag{1.51}$$

$$\lambda_n = n^2\pi^2 \tag{1.52}$$

Therefore, the solution to Equation (1.48) is assumed to be expressed as

$$T(x,t) = \sum_{n=1}^{\infty} c_n(t) e_n(x) \tag{1.53}$$

Substitution of Equation (1.53) into Equation (1.48) yields

$$L\left(\sum_{n=1}^{\infty} c_n e_n\right) = -\frac{\partial}{\partial t}\sum_{n=1}^{\infty} c_n(t) e_n(x) \tag{1.54}$$

$$\sum_{n=1}^{\infty} c_n \lambda_n e_n = \sum_{n=1}^{\infty}\left(-\frac{\partial c_n(t)}{\partial t}\right) e_n(x) \tag{1.55}$$

or

$$\frac{\partial c_n}{\partial t} = -\lambda_n c_n \tag{1.56}$$

which can be solved as

$$c_n = A_n e^{-\lambda_n t} \tag{1.57}$$

where $A_n$ is an unknown integral constant that should be determined from the initial condition. Combining Equation (1.57) with Equation (1.53) yields

$$T(x,t) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n t} e_n(x) \tag{1.58}$$

The initial condition of $T(x, 0) = f(x)$ at $t = 0$ is now substituted into Equation (1.58) as

$$f(x) = \sum_{n=1}^{\infty} A_n e_n(x) \tag{1.59}$$

As Equation (1.59) is the eigenfunction expansion of $f(x)$, its coefficient is

$$A_n = (f, e_n) = \int_0^1 f(x) e_n(x) dx \tag{1.60}$$

Thus the solution is expressed as

$$T(x,t) = \sum_{n=1}^{\infty} (f, e_n) e^{-\lambda_n t} e_n(x) \tag{1.61}$$

For example, for the following initial condition,

$$-\frac{\partial^2 T}{\partial x^2} = -\frac{\partial T}{\partial t} \tag{1.62}$$

$$u = 0 \quad \text{at} \quad x = 0,1$$

$$u = 1 \quad \text{at} \quad t = 0$$

the solution is explicitly expressed as

$$\begin{aligned} T(x,t) &= \sum_{n=1}^{\infty} (f, e_n) \exp(-\lambda_n t) e_n(x) \\ &= \sum_{n=1}^{\infty} (1, \sqrt{2}\sin n\pi x) \exp(-n^2\pi^2 t) \sqrt{2} \sin n\pi x \\ &= \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi} \exp(-n^2\pi^2 t) \sin n\pi x \end{aligned} \tag{1.63}$$

where (*f*, *g*) is the inner product defined as

$$(f, g) \equiv \int_0^1 f(x) g(x) dx \tag{1.64}$$

Figure 1.7 shows how the temperature decays.

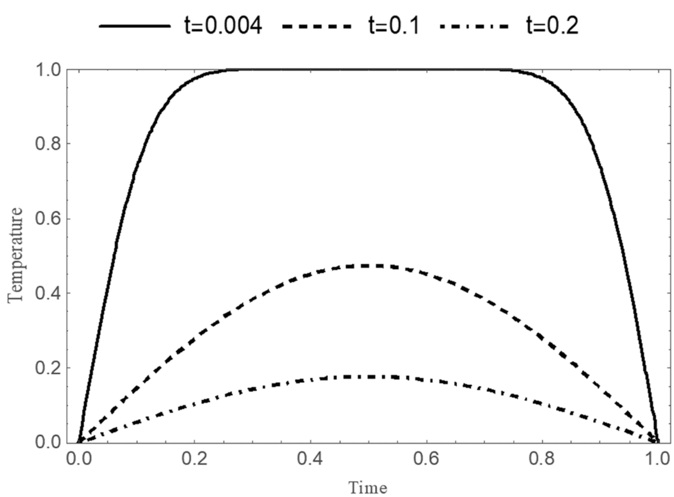


***FIGURE 1.7.*** *Temperature profile for the 1-D transient heat conduction problem at different times.*

Although the solution procedures presented in the previous three examples are for homogeneous materials, they can be used for heat transfer problems in composite materials in the subsequent chapters with some modifications.

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